Crossover to self-organized criticality in an inertial sandpile model

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(Received 26 June 1996)

We introduce a one-dimensional sandpile model that incorporates particle inertia. The inertial dynamics are governed by a parameter that, as it passes through a threshold value, alters the toppling dynamics in such a way that the system no longer evolves to a self-organized critical state. A range of mean-field theories based on a kinetic equation approach is presented that confirm the numerical findings. We conclude by considering the physical applications of this model, particularly with reference to recent experimental results. [S1063-651X(97)14403-8]

I. INTRODUCTION

The concept of *self-organized criticality* (SOC) was introduced by Bak, Tang, and Wiesenfeld [1] as a possible explanation for the common occurrence of scale-invariance in nature. To demonstrate this behavior, they introduced the *sandpile model*, a driven dissipative cellular automata whose dynamics are defined by local interactions. Despite the short-range dynamics, the system organizes itself into a nonequilibrium critical state with no finite correlation length and hence no characteristic length scale. A feature common to all SOC systems is that the sizes of fluctuations follow power-law distributions, a direct consequence of the scale invariance. However, comparisons with real systems have met with only partial success. Power laws were observed in a granular mixture when avalanches were initiated by watering the pile [2], but not in a pile of glass beads that was gradually tilted [3]. Adding grains individually to a conical sandpile only showed power laws for sufficiently small piles [4]. Recently, Frette *et al.* [5] performed experiments on one-dimensional piles of rice, and found power-law behavior only for grains with a sufficiently large aspect ratio. In light of these experiments, we believe it would be informative to construct a sandpile model with greater physical applicability, while hopefully retaining some of the interesting dynamics.

One important ingredient missing from the basic model is inertia. To the best of our knowledge, there have only been two attempts to construct an inertial sandpile model. Prado and Olami [9] chose to associate moving particles with a decrease in the local stability, and found SOC behavior only for small systems, in both one and two dimensions, thus giving an explanation for the results in [4]. Krug, Socolar, and Grinstein [8] gave a single measure of momentum for the entire cluster of moving particles in a one-dimensional system. They found that their inertia parameter needed to be fine tuned to zero for the system to become SOC.

In this paper, we consider an interesting way of incorporating inertia into the sandpile model. A full description of the model is given in Sec. II, but briefly we suppose that

moving particles only come to rest on those sites whose slopes are not too steep, where the definition of "too steep" is controlled by a parameter. Our main result is the crossover between SOC and non-SOC behavior as this parameter passes through a threshold value. The existence and value of this transition point has been confirmed by extensive numerical analysis, mean-field theories, and qualitative reasoning.

This paper is arranged as follows. Our model is defined, and some of its immediate consequences explored in Sec. II. In Sec. III, the results of numerical simulations are given for the simplest nontrivial sandpiles. These results are confirmed by the mean-field analysis given in Sec. IV, where a rate equation approach has been adopted. Finally, in Sec. V, we explain the nature of the transition threshold and discuss applying the model to real physical systems.

II. MODEL

A one-dimensional sandpile is defined by a set of integer heights h_i, $i=1,\ldots,L$, or equivalently by the local slopes $z_i=h_i-h_{i+1}$. The right-hand boundary is taken to be open, $z_L=h_L$, whereas the left-hand boundary is treated as closed, $z_0=0$. Particles are added sequentially to randomly chosen sites, increasing their height by one unit. In the limited local sandpile (LLS) model [10], a site i becomes unstable when z_i becomes greater than the *critical slope parameter z_c*. Any such unstable site will *topple*, z_c particles will leave site i, and will move to site $i+1$ (or leave the system if $i=L$), so

$$z_{i-1} \rightarrow z_{i-1} + z_c,$$

$$z_i \rightarrow z_i - 2z_c, \qquad (1)$$

$$z_{i+1} \rightarrow z_{i+1} + z_c,$$

with equivalent rules for toppling at boundaries. It is now possible for z_{i-1} and/or z_{i+1} to become unstable and topple, and an avalanche will begin. We call the series of the initial topplings at sites $i-1$, $i-2$, $i-3,\ldots$ the *back-avalanche*. A back-avalanche propagates to the first site $j<i$ with $z_j \leqslant 0$, where such sites are called *troughs*. In terms of the slopes before particle addition to site i, $z_{i-1}=1$ will also stop a back-avalanche. Some previous literature [11] refers to such instances as *slide events*. Topplings occur on a time

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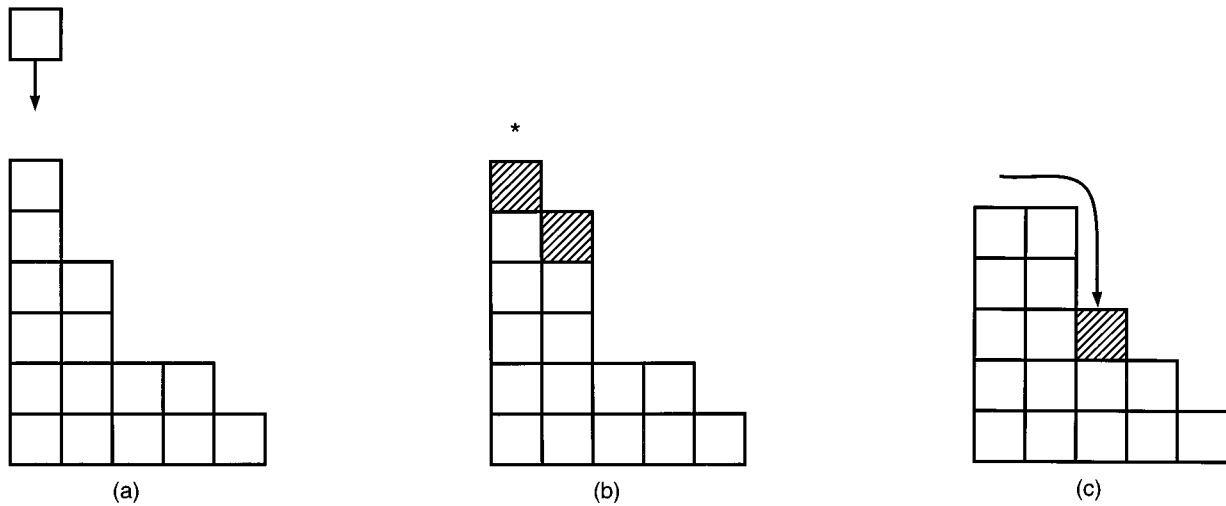
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FIG. 1. An example of the toppling process in the MILLS model, for $z_c=2$ and $z_{in}=3$. (a) Particle added to a site with critical slope. The site becomes unstable and topples. (b) The first toppling particle sticks to the adjacent site, giving it a local slope of 3. The site marked with an asterisk is still in the process of toppling. (c) The second particle slides through the site with slope 3 to stop at the next site. All sites in this region are now stable.

scale much smaller and separated from that of particle addition, in that no more particles are added to the pile until the avalanche is over and every site has become stable ($z_i \leq z_c$ $\forall i$).

As this model stands, the inertia of the toppling particles has been ignored, any momentum in the topple is assumed to be dissipated instantaneously from the sandpile as soon as the particles have moved. The previous attempts to incorporate inertia into the system [8,9] used quantities that evolved throughout the avalanche and hence introduced a form of memory into the system. In this paper, we consider a set of rules for toppling that requires just one extra, time-independent quantity, but nonetheless intuitively mimics inertia. We introduce the *minimum-slope inertial limited local sandpile* (MILLS) model, which has a second critical slope parameter, the *inertial threshold* z_{in}. Now when a site i topples, the particles will be deposited on the first site $j>i$ obeying

$$z_j < z_{in}, \tag{2}$$

or, if no such site exists, leave the system. This calculation is performed right to left for each unstable pile, and individually for each of the z_c particles involved in every toppling, recalculating the slopes as a particle comes to rest. An example is given in Fig. 1 for $z_c=2$ and $z_{in}=3$. Note that we have implicitly introduced a time frame for inertial effects which is much smaller than that of toppling events. This separation of time scales means that the time taken for an unstable site to begin to topple is much longer than that for the ensuing particle motion itself.

As a realistic physical system, this model does have some obvious drawbacks, the most notable being that a particle will be stopped just as easily if it has just toppled, or if it has rolled down a large region of slopes $\geq z_{in}$. We could, of course, make z_{in} dependent on each particle's previous motion, but it is the lack of inherent memory that makes much of the subsequent analysis possible. Furthermore, with this

implementation of inertia the sandpile takes a microscopically smooth, realistic shape which further justifies its study.

We can immediately make some general observations about the MILLS model. Since the particles toppling from an unstable site all move at least one step, the slope of the site to the immediate left will always increase by z_c independent of the value of z_{in}. Thus the back-avalanches in this model are identical to those in the standard model, and consequently troughs are still important as bounds of the left-hand edge of an avalanche. However, troughs no longer necessarily bound the right-hand edge of an avalanche as they did in the LLS model. Indeed, no expression involving a single site can now serve as a general right-hand bound, so the previous analysis of the standard model based on troughs [6,8] cannot be extended to this model.

For $z_c=1$, the sandpile soon reaches a trivial steady state with $z_i=1$ $\forall i$ independently of z_{in}. Inertial effects are now indistinguishable from topplings, and added particles move downslope by either mechanism and leave the system in an unaltered state. Varying z_{in} will change the time taken for the particle to reach the right-hand boundary, but only for $z_c \geq 2$ can z_{in} have any influence on the nature of the critical state.

If $z_{in}=-\infty$, inertial effects will dominate, and all toppling particles will immediately leave the system, as if the sandpile has become infinitely slippery. This is equivalent to the case $r=0$ for the model in [8]. Since sites with negative slope are rare, $z_{in} \leq 0$ will have a similar effect to $z_{in}=-\infty$. For $z_{in} \leq z_c$, no particle will come to rest on a site with critical slope and so no sites to the right of the initial topple can become unstable. As a result, there will be no forward toppling, and the avalanche will consist solely of the back-avalanche.

It may appear that implementing inertia in this manner could allow for a large cluster of sliding particles to come to rest simultaneously on the same site, say a site i with z_i large and negative. It is easy to prove, however, that at most z_c particles can stop on a site during any one step in the top-

pling process. To see this, first realize that a back-avalanche consists of one topple per avalanche time step, so to get more than z_c particles moving at once we must have $z_{in} > z_c$. Furthermore, a site $k < i$ must topple, and subsequently allow at least one particle to slide through it, i.e.,

$$z_k - z_c \geq z_{in}, \qquad (3)$$

so for this sequence of events to occur requires $z_k > 2z_c$. Although site $k+1$ toppling could contribute to such a large slope, having $z_{k+1} > z_c$ on the previous step would simultaneously require $z_k \leq 0$, except at the left-hand boundary of the avalanche, as closer analysis soon reveals. It should now be apparent that for any site to gain more than z_c particles during any one avalanche step, there must already exist a site in the system to which this has already happened. Since particle addition cannot create such a site, we must conclude that they never occur. Intuitively, this result implies that inertial effects serve only to broaden the shape of the avalanche.

If $z_{in} = \infty$, inertial effects will be so weak that toppling particles will never move by more than one site, and we just have the LLS model. In fact, from the result given in the previous paragraph, it should be apparent that the maximum slope that a site can reach is $2z_c$, so if $z_{in} \geq 2z_c$, inertial effects become redundant, and the MILLS model reduces to the standard model.

In [5] it was suggested that it was the shape of the particles that determined the nature of the critical pile. Although any sandpile model is oversimplified compared to real piles of any substance, for instance, in failing to allow for the particles to overlap and for ignoring any variation in particle dimensions, it is still instructive to consider qualitatively how the two parameters z_c and z_{in} might relate to actual systems. Suppose the particles all have an aspect ratio α. A vertical stack consisting of rounded particles will clearly be much less stable than a stack of flat particles, so z_c will decrease as α decreases. The parameter z_{in} should also decrease with α, since rounder particles will roll more easily and each site will be narrower and hence easier to traverse. Indeed, varying α should cause z_{in} to change faster than z_c does, so we can see that the ratio z_{in}/z_c will increase with α. A more precise analysis in this manner is possible, but any such accuracy is lost within the artificial framework common to all sandpile models.

In summary, the MILLS model differs from the LLS model by the extra parameter z_{in}. For $z_{in} \geq 2z_c$ we just get the LLS model, for $z_{in} \leq z_c$ the toppling process consists of just the back-avalanche, and for $z_{in} \leq 0$ the majority of sliding particles leave the system immediately after toppling.

III. RESULTS

A feature common to all SOC models is the existence of power laws. This is only strictly true for infinite systems, however. Finite systems, such as those carried out on a computer, tend to include a finite-size cutoff. The one-dimensional sandpile model is somewhat unique in this respect as the finite-size effects come to dominate the system, for reasons that are not yet fully understood [6]. Even simulations for the standard model with $L > 50\,000$ have failed to

exhibit true power-law behavior [7]. In this paper, we follow Krug, Socolar, and Grinstein [8] in identifying scale invariance with a broadening of the avalanche size distribution with the system size. This is a valid approach, since if the critical state were instead governed by exponentially decaying correlations, then increasing the system size far beyond the correlation length could not alter the avalanche size, and so there would be no such broadening. Two examples of drop-size frequency distributions are given in Fig. 2, where the drop size is defined as the number of particles to leave the pile as the result of a single particle addition. The distribution broadens for $z_{in} = 2z_c$ when the system size is doubled, which is clearly not the case for $z_{in} = 0$; thus the system is not SOC for $z_{in} = 0$. It is obviously important to find a value of z_{in} between these two extremes at which the transition between SOC and non-SOC behavior occurs.

The distribution of drop sizes is not always a useful measure of avalanche size, since for $z_{in} = z_c$ the maximum drop size is just $z_c + 1$, as the following analysis demonstrates. Suppose an avalanche is started by the addition of a particle onto a site i with critical slope, so that before toppling we have $z_i = z_c + 1$ and $z_{i-1} \leq z_c - 1$. When site i topples, z_i will decrease by at least z_c, possibly more if any particles stick to $i+1$, so now $z_i \leq 1$ and $z_{i-1} \leq 2z_c - 1$. Presuming now that $i-1$ topples, all but possibly one particle will stick to i, giving $z_{i-1} \leq 0$. Even if $i-2$ now topples, no particles can slide past $i-1$, and so now $z_{i-2} \leq 0$. From now on this is all that happens; toppled particles neither slide nor cause any further topples. The avalanche will eventually end with at most $z_c + 1$ particles from the leading edge of the avalanche moving beyond site i and potentially contributing to the drop size. More qualitatively, toppling is minimal for $z_{in} \leq z_c$, and the bulk of the sliding is limited to within the avalanche for $z_{in} \geq z_c$, so for $z_{in} = z_c$ we should expect particle transport to be low. Less commonly considered measures of avalanche size, such as the total number of topples involved, must be used in this case.

For $z_{in} = 1$ and $z_c = 2$, using the drop-size distributions to test for SOC behavior becomes ambiguous. Although there is broadening for small system sizes, the plots for larger systems are inconclusive, and cannot be used to differentiate between true scale invariance or just a large correlation length. An alternative does exist, since when $z_{in} \leq z_c$ the avalanche size is bounded above by the distance between troughs, so a system with a finite concentration of troughs must have a finite correlation length and hence cannot be SOC. The trough density, being just a single value, is much less susceptible to noise, and can be measured reliably for larger L. The distribution of trough densities for $z_c = 2$ and for L up to 2048 is given in Fig. 3. That $z_{in} = 1$ is not SOC is now evident, as the trough density tends to a finite value, although this value is small, corresponding to a large correlation length, which explains the observed broadening of the dropsize distribution in small systems. For $z_{in} = z_c = 2$, the trough density approaches zero with system size faster than for any other value of z_{in}, corresponding to a singular correlation length and SOC behavior. These results are borne out by the mean-field analysis in Sec. IV.

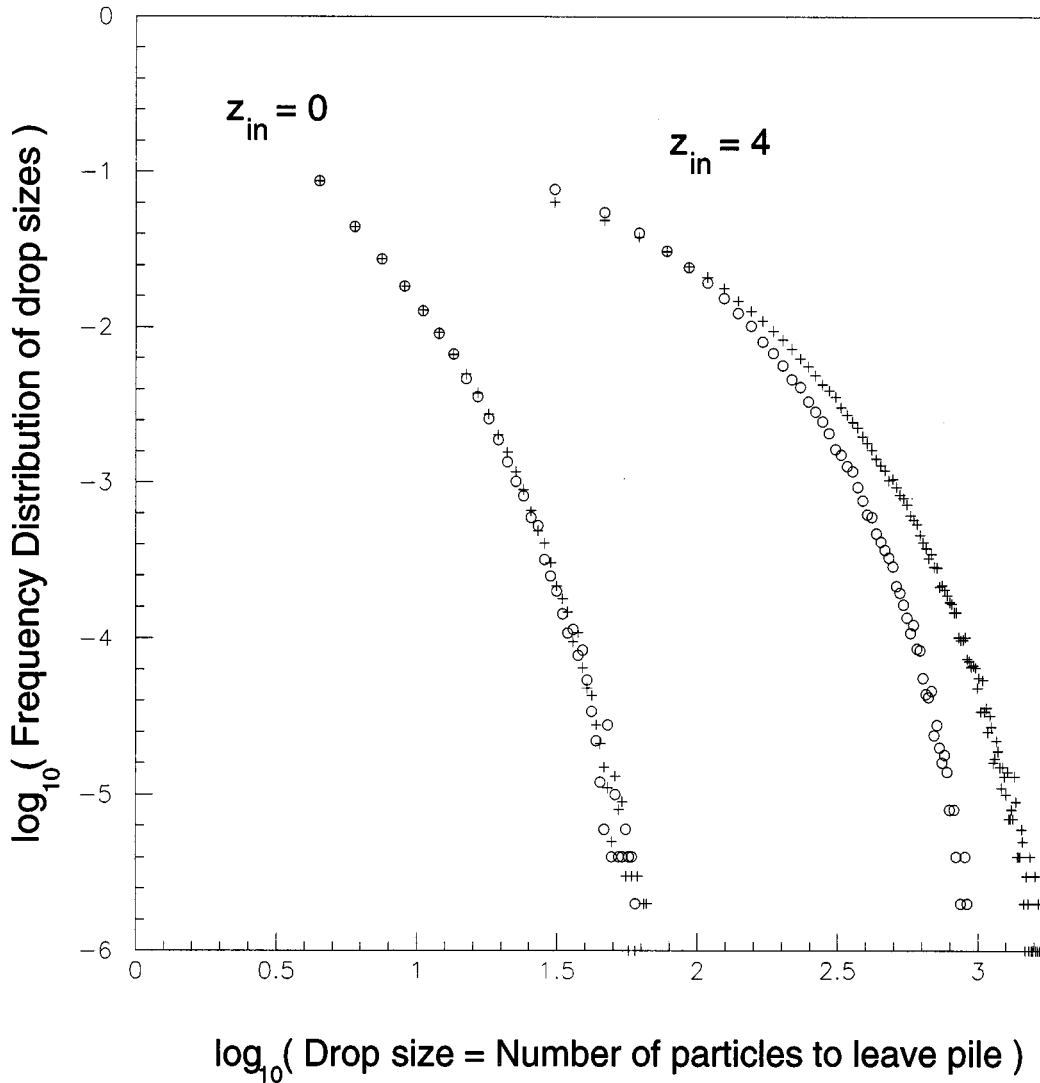Another useful measure to consider is the average slope S,

FIG. 2. Distribution of drop sizes for $z_{in}=4$ and 0, and for two different system sizes, with $z_c=2$. Circles correspond to $L=512$, crosses to $L=1024$.

$$S = \frac{1}{L} \sum_{i=1}^{L} z_i, \qquad (4)$$

where $\tan^{-1}(S)$ is the *angle of repose*. The variation of S with L and z_{in} is given in Fig. 4. For $z_{in} < 2$ the lines converge on values in the range $1 < S < 1.2$. It is already known [7] that for $z_{in} = 4$ the slope for $L \rightarrow \infty$ is $S = \frac{3}{2}$. Comparing this line to those for other z_{in} in Fig. 4, it could be judged that the curves for $z_{in}=2$ and $z_{in}=3$ both tend to $S=2$ in a similar manner. Since convergence is slow, however, verifying these asymptotic limits is impossible from this data alone. Taking logarithmic plots to project the lines further, as was done successfully for the standard model [7], fails here as it gives reasonable straight line fits for a range of limiting slopes. Instead, a rough prediction of the asymptotic slope for $z_{in}=2$ is given by the following qualitative argument.

For $z_{in} \leq z_c$, particles leaving the pile cannot cause the rightmost site $i=L$ to topple, although if $z_L < z_{in}$ they will stick there and increase z_L. Once z_L is in the range $z_{in} \leq z_L \leq z_c$, all particles will slide over L, so z_L is fixed until the end of the current avalanche. The same thing will

then happen for site $L-1$, then $L-2$, and so on. Thus the effect of any avalanche reaching the right-hand boundary is to leave behind a cluster of sites with slopes in the interval $[z_{in}, z_c]$. Furthermore, the slopes in this region are stable under subsequent particle addition on any site to its left. Similar clusters should grow throughout the pile for similar reasons, although their right-hand edges will not be fixed. This has special significance for $z_{in}=z_c$, since the clusters of sites will all have slope z_c, and S is expected to tend to a value at least close to z_c. Given that this system becomes dominated by sites of critical slope, the z_c+1 particles that slide from any avalanche should all leave the pile; thus the modal drop size should also be the maximum one, z_c+1. This has been confirmed by the numerical studies.

IV. MEAN-FIELD ANALYSIS

Christensen and Olami [12] successfully formulated a random nearest-neighbor mean-field model for critical height sandpiles. Directly extending their procedure to critical slope
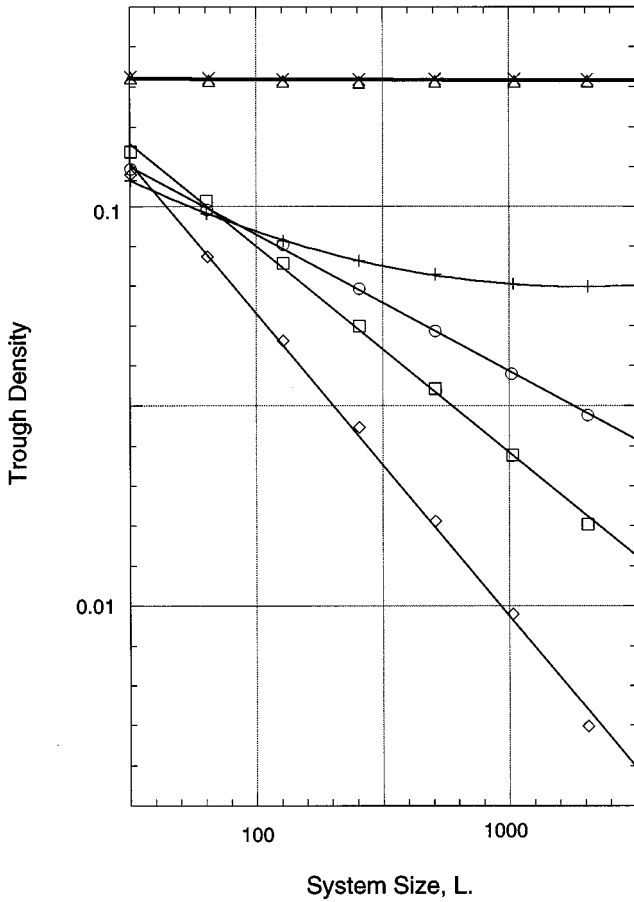
FIG. 3. Trough densities [(number of troughs)/L] for $z_c=2$ against system size. Circles refer to $z_{in}=4$, squares to $z_{in}=3$, diamonds to $z_{in}=2$, plus signs to $z_{in}=1$, triangles to $z_{in}=0$, and crosses to $z_{in}=-\infty$.

FIG. 4. Average slopes for $z_c=2$ against system size. Circles refer to $z_{in}=4$, squares to $z_{in}=3$, diamonds to $z_{in}=2$, plus signs to $z_{in}=1$, triangles to $z_{in}=0$, and crosses to $z_{in}=-\infty$.

models has proved to be intractable, but the principle of taking random neighbors in an infinite system can still be applied to the MILLS model with some success. Utilizing this approach is nontrivial, however, as the dynamics of the system change significantly as z_{in} is varied. In this section we present a range of models for $z_{in}\leqslant 4$ with $z_c=2$, comparing their predictions to the numerical findings. All these models are cast in terms of the slope distribution $\{S_n(t)\}$, where each $S_n(t)$ is defined to be the proportion of sites in the system at time t with slope n. For convenience, we also define

$$S_T(t)=\sum_{i=-\infty}^{0} S_i(t), \qquad (5)$$

which is the trough density. As these slopes represent the sandpile in its stable state, the maximum value any z_i can take is two and

$$S_T+S_1+S_2=1. \qquad (6)$$

The simplest case to analyze is $z_{in}=-\infty$, where any avalanche propagates back to the first trough, and all toppled particles are removed from the system. Particle addition alone has the effect of increasing the slope of a randomly chosen site from n to $n+1$,
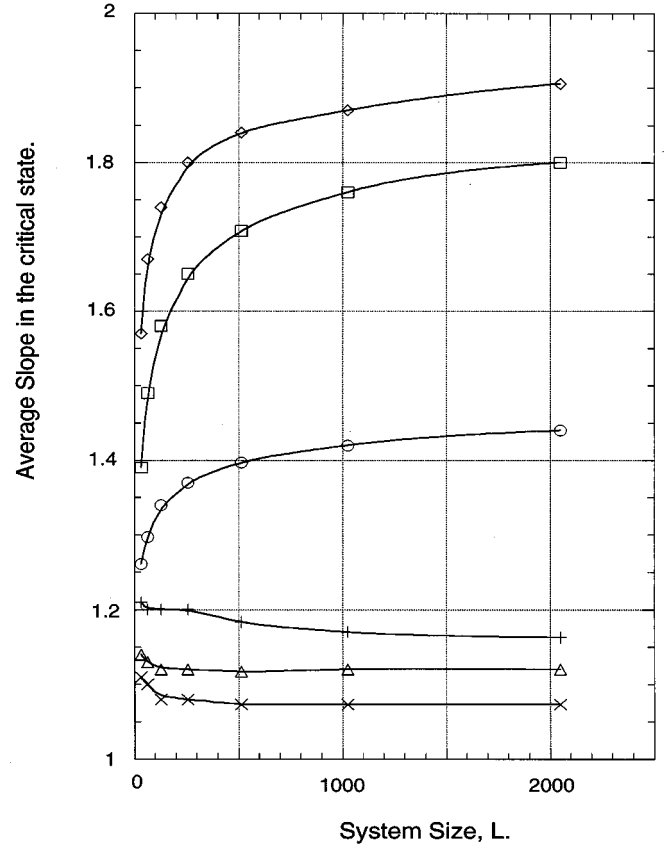
$$S_n\rightarrow S_n-\frac{1}{L}, \qquad (7)$$

$$S_{n+1}\rightarrow S_{n+1}+\frac{1}{L}, \qquad (8)$$

and in a similar manner decreasing the slope of another randomly chosen site by one. Troughs are assumed to be primarily sites of slope zero, so increasing the slope of a trough will give a site of slope 1, whereas the effects of decreasing the slopes of troughs are ignored.

When a particle is added to a site of slope two, Eqs. (7) and (8) must be replaced by equivalent rules for the whole of the subsequent sequence of topples. There are two cases to consider—if the site to the immediate left (here chosen at random) has slope 1, a slide event occurs, removing two particles from the system but leaving $\{S_n(t)\}$ unchanged. Otherwise, we obtain a full avalanche which only alters the slopes at its leading edge (where the particle was added) and its trailing edge (at the trough that halted the toppling). The net result of both these edge effects is the lose a trough and a site of slope two, and to gain two sites of slope 1. It is now straightforward to write down rate equations for $\{S_n(t)\}$, with the time scale normalized to L particle additions per unit t,

$$\frac{dS_2}{dt} = -2S_2 + S_1 + S_1 S_2, \qquad (9)$$

$$\frac{dS_1}{dt} = -2S_1 + S_T + 3S_2 - S_1 S_2, \qquad (10)$$

$$\frac{dS_T}{dt} = -S_T + S_1 - S_2. \qquad (11)$$

These equations can be justified directly—for instance, the first term on the right-hand side of Eq. (10) corresponds to particle addition onto or adjacent to a site of slope 1, the second term is for a particle added to a trough, the third term accounts for particle addition adjacent to or onto a site of slope 2, where the latter instance invokes an avalanche, and finally the fourth term subtracts out trivial slide events. In practice, however, it is simpler to derive these equations by writing down all nine possible combinations of pairs of sites, i.e., (TT) $(T1)$ $(T2)$ $(1T)$ (11) (12) $(2T)$ (21) and (22), then computing directly the effects of particle addition to the right-hand site and weighting them accordingly.

Equations (9)–(11) converge on the stable solution $(S_T, S_1, S_2) = (\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$, in good agreement with the numerical values $(0.18, 0.48, 0.34)$. Since the influence of sites with strictly negative slope have been ignored, this mean-field theory actually covers the range of all $z_{\text{in}} \leq 0$, and indeed the agreement still holds up to $z_{\text{in}} = 0$. For $z_{\text{in}} = 1$ the avalanching particles can no longer be ignored, and terms need to be added to account for where they come to rest. The expected number of particles brought into motion during an avalanche depends on the state of the site to the immediate left of the site where the first particle was added. If this site has slope less than two, then just the one site topples. Otherwise, the number of sites that topple is half the average distance between troughs. In this random neighbor model, the probability that two nearest troughs are a distance r apart is $(1 - S_T)^r S_T$. Consequently, half the mean distance between troughs is $(1 - S_T)/2S_T$, and the total number of sites that topple during an avalanche is on average $S_T + S_1 + S_2[(1 - S_T)/2S_T]$. Each toppled site releases two particles, so we will need a factor of 2. Furthermore, since avalanches are initiated by particle addition to sites of slope 2, we will also need a factor of S_2. Thus, the expected number of toppling particles per unit time is

$$\lambda(t) = S_2 \left(2(1 - S_2) + \frac{1 - S_T}{S_T} S_2 \right). \qquad (12)$$

Each of these $\lambda(t)$ particles will stop on a trough, creating a site with slope 1. This will also cause the site to the immediate left to decrease its slope by one. Moreover, we know that the particle passed over this site, so it cannot be a trough. Therefore it must have slope 1 or 2 with probability $S_1/(1 - S_T)$ or $S_2/(1 - S_T)$, respectively. Since we are ignoring any correlations between where a particle starts moving and where it stops, all we need to do is to add extra terms to the right hand sides of Eqs. (9)–(11) to account for the changes in the slopes caused by the $\lambda(t)$ particles coming to rest,

$$\frac{dS_2}{dt} = -2S_2 + S_1 + S_1 S_2 - \lambda \left(\frac{S_2}{1 - S_T} \right), \qquad (13)$$

$$\frac{dS_1}{dt} = -2S_1 + S_T + 3S_2 - S_1 S_2 + \lambda \left(1 + \frac{S_2 - S_1}{1 - S_T} \right), \qquad (14)$$

$$\frac{dS_T}{dt} = -S_T + S_1 - S_2 + \lambda \left(\frac{S_1}{1 - S_T} - 1 \right). \qquad (15)$$

These equations now converge to $(S_T, S_1, S_2) \approx (0.09, 0.65, 0.26)$, whereas numerically the slope distribution is $(0.07, 0.68, 0.25)$. Note that the trough density S_T is small but non-zero, corresponding to a large but finite correlation length. Thus, as noted in Sec. III, large system sizes are required to demonstrate that this system is not SOC.

For $z_{\text{in}} = 2$, the interesting nature of the avalanche requires a special set of rate equations, which can be derived in a similar manner to before by considering the consequences of particle addition to a pair of sites with arbitrary slopes,

$$\frac{dS_2}{dt} = S_T(S_1 - 2S_2) + S_1^2 + \mu \left(\frac{S_1}{1 - S_2} - 1 \right), \qquad (16)$$

$$\frac{dS_1}{dt} = -2S_1^2 + S_T(3S_2 - S_1 + S_T) + \mu \left(1 + \frac{S_T - S_1}{1 - S_2} \right), \qquad (17)$$

$$\frac{dS_T}{dt} = -S_T + S_1 - S_2 S_1 - \mu \left(\frac{S_T}{1 - S_2} \right), \qquad (18)$$

where $\mu(t) = S_2(2 + S_2)$ is the expected number of particles that emerge to the right of the particle addition. Although these equations admit the steady solution $(S_T, S_1, S_2) \to (0, 0, 1)$ with $S_1 \sim 1 - S_2$ and $S_T \sim 0$, cubic corrections are required for it to be stable. Numerically, the slopes are $(0.01, 0.08, 0.91)$ for $L = 2048$ but, as mentioned in Sec. III, the asymptotic slope appears to be $S = 2$, which demands the slope distribution $(0, 0, 1)$.

For $z_{\text{in}} = 3$, all the sites involved in an avalanche resolve in a highly nontrivial way, so calculating rates between the states before and after an avalanche becomes impossible. Instead, we introduce the following *local slope argument* to predict the slope distribution in the critical state. Within an avalanche, a toppled site deposits a pair of particles to its right, which, after any inertial motion, settle and possibly cause further toppling. Keeping our random neighbor approach, the expected number of further topples caused takes the same constant value E_t throughout the avalanche. If $E_t < 1$ then avalanches would die out exponentially quickly, particles will not be transported from the pile, and so the sandpile will build up, increasing the average slope and hence also E_t. If $E_t > 1$, avalanches would grow exponentially and large numbers of particles would leave the pile, decreasing E_t. Although transport from the pile is not actually catered for by this model, we infer that $E_t = 1$ in the critical state, so finding an expression for E_t in terms of $\{S_n(t = \infty)\}$ gives an additional constraint which might help to fix the slope distribution.

This local slope argument has several deficiencies, apart from the obvious drawbacks of the random neighbor approach. Back-avalanches have been ignored, so this method will certainly break down for $z_{in} \lesssim 2$. The effects of single-particle addition has not been accounted for, which will further exasperate the situation when S_2 is small. Although this gives a limited range of applications for this model, it has the advantage of being conceptually and mathematically simple. For instance, in the standard model $z_{in} = 4$, adding two particles to any site with a slope greater than zero will cause one further topple, so $E_t = 1 - S_T$. Requiring that this equals 1 gives $S_T = 0$, as expected.

For $z_{in} = 3$, consider the effect of adding the particles to two adjacent sites, i and $i+1$. If $z_i < 2$, then both particles would stick to i, giving the site a slope $z_i + 2$, which will topple if $z_i = 1$. However, if $z_i = 2$ then the last particle added will slide onto the adjacent site, resulting in $z_{i+1} \rightarrow z_{i+1} + 1$ and z_i remaining unchanged. Thus the expected number of further topples is

$$E_t = S_1 + S_2^2. \tag{19}$$

Setting this equal to 1 and employing Eq. (6), we obtain an equation for the trough density,

$$S_T = S_2(S_2 - 1). \tag{20}$$

By definition, all the S_i must lie in the range $[0,1]$. However, Eq. (20) is strictly negative for $0 < S_2 < 1$, so for consistency we must have $S_2 = 0$ or 1. Avalanches could never begin in a system with $S_2 = 0$, so we conclude that $(S_T, S_1, S_2) = (0, 0, 1)$, in agreement with the numerical work.

V. CONCLUSIONS AND DISCUSSION

It has been demonstrated that the MILLS model with $z_c = 2$ displays self-organized criticality for systems with $z_{in} \geq z_c$, but SOC is lost when $z_{in} < z_c$. The SOC state for $z_{in} \geq z_c$ was identified by the broadening of the drop-size distribution with the system size, whereas for $z_{in} < z_c$ the existence of a finite correlation length was found both numerically and from the mean-field theories. Preliminary numerical results for z_c up to 4 demonstrate that the transition point $z_{in} = z_c$ holds more generally. We postulate that for all $z_c > 1$ the model is SOC only when $z_{in} \geq z_c$, and now argue why this should be so.

An essential requirement of any SOC system is that it is governed by local driving forces [1]. For $z_{in} > z_c$, inertial effects only come into play inside an avalanche, since it is only within an avalanche that sites with slopes greater than z_c can occur. Thus the couplings remain short range, and SOC behavior is preserved. For $z_{in} < z_c$, however, particles can slide far beyond the avalanche, so interactions become nonlocal, and the system ceases to be SOC. We might naively expect that a system with $z_{in} = z_c$ will also fail to be SOC, especially since the system becomes dominated by sites of slope z_c, which should allow particles to slide through large distances. However, the exact analysis in Sec. III proved that only a tiny proportion of particles move beyond the avalanche. In fact, the vast majority move by just one site, which is certainly a local driving force, so we can conclude that a system with $z_{in} = z_c$ is SOC.

In the experiments carried out on piles of rice [5], power-law behavior was only observed for rice with sufficiently large aspect ratio. In Sec. II it was argued that z_{in}/z_c should increase with the particle aspect ratio. Thus, if the aspect ratio is increased such that z_{in} becomes greater than z_c, the MILLS model gives a possible explanation for the appearance of power laws—the system has crossed over into the SOC regime. If this were indeed the case, we must assume that the finite-size effects which destroyed true power-law behavior in our numerical simulations do not have the same effect in the experimental setup. However, many non-SOC systems also exhibit power laws, and an alternative model has recently been proposed by Newman and Sneppen [13]. Here, they claimed that a SOC approach is invalid when the power laws have characteristic exponents in the vicinity of 2, as was observed for the rice piles. Their model is instead driven by external noise, but still predicts a crossover from power-law behavior.

Regardless of its experimental applications, the MILLS model represents an interesting addition to the range of sandpile systems already studied, in that it provides an inertial sandpile model that is SOC for a broadband of parameter space—that is, without the need for fine tuning—and independent of the system size. Extending this model to higher dimensions or to critical height sandpiles should in principle be straightforward, but since our original motivation was to try and account for the results from the rice pile experiments, we chose to base our model on that system.

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